Week 9 - Wednesday

COMP 2230

Last time

- Multiplication rule
- Addition rule
- Started pigeonhole principle

Questions?

Assignment 4

Logical warmup

- Recall that a chessboard has 8 rows and 8 columns
- You start on the upper left square of a chessboard
- You are only allowed to move downwards or to the right, and can only move one square at a time
- How many routes are there to the lower right square?

Pigeonhole Principle

Pigeonhole principle

- If n pigeons fly into m pigeonholes, where $n > m$, then there is at least one pigeonhole with two or more pigeons in it
- More formally, if a function has a larger domain than co-domain, it cannot be one-to-one
- We cannot say exactly how many pigeons are in any given holes
- Some holes may be empty
- But at least one hole will have at least two pigeons

Pigeonhole examples

- A sock drawer has white socks, black socks, and red argyle socks, all mixed together,
- What is the smallest number of socks you need to pull out to be guaranteed a matching pair?

- Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- If you select five distinct elements from A, must it be the case that some pair of integers from the five you selected will sum to 9?

Generalized pigeonhole principle

- If n pigeons fly into m pigeonholes, and for some positive integer k, $n > km$, then at least one pigeonhole contains $k + 1$ or more pigeons in it
- Example:
  - In a group of 85 people, at least 4 must have the same last initial

Combinations

Subsets of sets

- How many different subsets of size r can you take out of a set of n items?
  - Subset of size 3 out of a set of size 5?
  - Subset of size 4 out of a set of size 5?
  - Subset of size 5 out of a set of size 5?
  - Subset of size 1 out of a set of size 5?
- This is called an r-combination, written $\binom{n}{r}$

Permutations and combinations

- In r-permutations, the order matters
- In r-combinations, the order doesn't
- Thus, the number of r-combinations is just the number of r-permutations divided by the possible orderings

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

Combinations example

- How many ways are there to choose 5 people out of a group of 12?

- What if two people don't get along? How many 5 person teams can you make from a group of 12 if those two people cannot both be on the team?

Poker examples

- How many five-card poker hands contain two pairs?
- If a five-card hand is dealt at random from an ordinary deck of cards, what is the probability that the hand contains two pairs?

r-combinations with repetitions

- What if you want to take r things out of a set of n things, but you are allowed to have repetitions?
- Think of it as putting r things in n categories
- Example: $n = 5, r = 4$

1	2	3	4	5
x		xx	x	

- We could represent this as x||xx|x|
- That's an r x's and $n - 1$ |'s

r-combinations with repetitions

- So, we can think of taking an r-combination with repetitions as choosing r items in a string that is $r + n - 1$ long and marking those as x's
- Consequently, the number of r-combinations with repetitions is

$$\binom{r+n-1}{r}$$

Example

- Let's say you grab a handful of 10 Starbursts
- Original Starbursts come in
  - **Cherry**
  - **Lemon**
  - **Strawberry**
  - **Orange**
- How many different handfuls are possible?
- How many possible handfuls will contain at least 3 cherry?

Handy dandy guide to counting

- This is a quick reminder of all the different ways you can count things:

	Order Matters	Order Doesn't Matter
Repetition Allowed	n^k	$\binom{k+n-1}{k}$
Repetition Not Allowed	$P(n,k)$	$\binom{n}{k}$

Binomial Theorem and Expected Value

Three-Sentence Summary

Binomial Matters

Pascal's Triangle

- Hopefully, you're all familiar with Pascal's Triangle, the beginning of which is:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

- If we number rows and columns starting at 0, note that the value of row n, column r is exactly $\binom{n}{r}$

Pascal's Formula

- Pascal's Triangle works because of Pascal's Formula:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

- We can show its truth:

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$$

$$= \frac{n!\,r}{r!\,(n-r+1)!} + \frac{n!\,(n-r+1)}{r!\,(n-r+1)!} = \frac{n!\,(n+1)}{r!\,(n-r+1)!} = \frac{(n+1)!}{r!\,(n+1-r)!}$$

Binomial Theorem

- $a + b$ is called a **binomial**
- Using combinations (or Pascal's Triangle) gives an easy way to compute $(a + b)^n$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

- We could prove this by induction, but you probably don't care

Binomial Example

- Compute $(1 - x)^6$ using the binomial theorem

Ticket Out the Door

Upcoming

Next time…

- Conditional probability

Reminders

- **Keep working on Assignment 4**
  - **Due Friday**
- Read 9.9
  - Prepare a three-sentence summary
  - Extra credit if you get called on